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Continuous Symmetries of Three-Dimensional Diffusion Equations

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Lie transformation groups are given which leave the three-dimensional linear diffusion equation invariant, with and without chemical reactions. We show how similarity solutions and conserved currents can be obtained with the help of these groups. We apply these methods to nonlinear three-dimensional diffusion equations which can be exactly linearized by nonlinear transformations.

1. INTRODUCTION

Lie transformation groups which leave invariant the one-dimensional diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2$ have been studied by several authors (Blumen and Cole, 1974; Harrison and Esterbrook, 1971; Steeb, 1978a, b; Steinberg and Wolf, 1981; Steeb and Strampp, 1982). Continuous symmetry groups of given field equations are helpful for obtaining similarity solutions (Blumen and Cole, 1974) and conserved currents (Steinberg and Wolf, 1981; Steeb and Strampp, 1982).

In the present paper we give Lie transformation groups which leave invariant the three-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{1}$$

The diffusion constant D, which is assumed to be constant, is included in the time t according to the transformation $t \to t/D$. Moreover, we show how the knowledge of the symmetry groups can be used for obtaining similarity solutions and conserved currents. Since a class of nonlinear diffusion equations can be transformed via a nonlinear transformation into the linear diffusion equation we are able to construct similarity solutions and conserved currents of this class of nonlinear diffusions. This class of nonlinear

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diffusion equations has been studied in one dimension by several authors. With the help of an example we demonstrate this approach.

In Section 2 we consider for the sake of completeness the one-dimensional diffusion equation.

In Section 3 the Lie transformation groups and their infinitesimal generators are given for the three-dimensional diffusion equation which leave the diffusion equation invariant. Moreover, we study three-dimensional diffusion equations where chemical reactions are included.

The Lie algebra which is associated with the infinitesimal generators is investigated in Section 4.

Similarity solutions are derived in Section 5.

Section 6 is devoted to the diffusion equation and Lie–Bäcklund transformations. Here we use the jet bundle formalism. This approach is briefly described.

Conserved currents of the diffusion equation are studied in Section 7.

Finally, we consider in Section 8 a class of nonlinear diffusion equations which can be linearized via a nonlinear transformation.

2. ONE-DIMENSIONAL DIFFUSION EQUATION

We briefly describe the one-dimensional diffusion equation and its symmetry groups. We do not give the symmetry groups, but we give the infinitesimal generators (vector fields). With the help of a Lie series we can obtain the symmetry group from the infinitesimal generator. The one-dimensional diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2$ is invariant under the following infinitesimal generators:

$$
\begin{gathered}
X = \frac{\partial}{\partial x}, \qquad T = \frac{\partial}{\partial t} \\
V = u\frac{\partial}{\partial u}, \qquad S = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \\
G = t\frac{\partial}{\partial x} - \frac{xu}{2}\cdot\frac{\partial}{\partial u} \\
P = xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{4} + \frac{t}{2}\right)u\frac{\partial}{\partial u}
\end{gathered} \tag{2}
$$

For the definition of the invariance of a partial differential equation we refer to Bluman and Cole (1974). In Section 6 we extend this definition. The vector fields given above form a basis of a non-Abelian Lie algebra.

The meaning of the generators is as follows: X represents translation in x and T translation in t. V represents the field scale change and S the scale change with respect to x and t. G represents the Galilean transformation and P is associated with the projective transformation. The generators given above lead via the mapping (Lie series)

$$(x, t, u) \to \exp(\varepsilon K)(x, t, u) \tag{3}$$

to the Lie transformation groups. ε is the group parameter and K the generator. Lie–Bäcklund transformation groups will be studied in the three-dimensional case.

A vector field which leaves the diffusion equation invariant and leads to a Lie transformation group has been omitted so far, namely, $U = \partial/\partial u$. This vector field leads to an infinite hierachy of infinitesimal generators which leave the diffusion equation invariant. This is due to the fact that the commutator of two generators of symmetry groups is again a generator of a symmetry group. For example,

$$[U, P] = -\left(\frac{x^2}{4} + \frac{t}{2}\right)\frac{\partial}{\partial u} \tag{4}$$

The right-hand side is a generator of a symmetry group. Taking the commutator of the vector field given by the right-hand side of equation (4) and P we find a further vector field for which the diffusion equation is invariant. The procedure can be carried out up to infinity. We notice that $f(x, t) = -(x^2/4 + t/2)$ is a solution to the diffusion equation. In general, we can easily formulate the following theorem.

Theorem. Let $f(x, t)\partial/\partial u$ be a vector field. Assume that f satisfies the diffusion equation. Then the diffusion equation is invariant under the vector field $f(x, t)\partial/\partial u$.

We mention that the converse is also true. The proof of the theorem will be given in Section 6 within the jet bundle formalism. The so-called diffusion polynomials [called heat polynomials in the literature (Widder, 1975)] can be found as follows. Consider the symmetry generators G and U. Then the commutators

$$\begin{gathered}[G, U]\\ [G,[G, U]]\\ \vdots\end{gathered} \tag{5}$$

and so on yield the diffusion polynomials. By a straightforward calculation we find

$$[G, U] = \frac{x}{2} \cdot \frac{\partial}{\partial u}$$
$$[G, [G, U]] = \left(\frac{t}{2} + \frac{x^2}{4}\right) \frac{\partial}{\partial u} \tag{6}$$
$$[G, [G, [G, U]]] = \left(\frac{3tx}{4} + \frac{x^3}{8}\right) \frac{\partial}{\partial u}$$
$$\vdots$$

Thus the first few polynomials are given by

$$p_1(x, t) = \frac{x}{2}, \qquad p_2(x, t) = \frac{t}{2} + \frac{x^2}{4}, \qquad p_3(x, t) = \frac{3tx}{4} + \frac{x^3}{8} \tag{7}$$

As described above the diffusion polynomials are solutions to the diffusion equation.

3. THREE-DIMENSIONAL CASE

Consider now the three-dimensional case. The three-dimensional diffusion equation given by equation (1) is invariant under the following vector fields:

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y}, \qquad Z = \frac{\partial}{\partial z}, \qquad T = \frac{\partial}{\partial t}$$
$$V = u\frac{\partial}{\partial u}, \qquad S = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + 2t\frac{\partial}{\partial t}$$
$$G_1 = t\frac{\partial}{\partial x} - \frac{xu}{2} \cdot \frac{\partial}{\partial u}, \qquad G_2 = t\frac{\partial}{\partial y} - \frac{yu}{2} \cdot \frac{\partial}{\partial u}, \qquad G_3 = t\frac{\partial}{\partial z} - \frac{zu}{2} \cdot \frac{\partial}{\partial u} \tag{8}$$
$$R_{12} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, \qquad R_{23} = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \qquad R_{31} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$
$$P = xt\frac{\partial}{\partial x} + yt\frac{\partial}{\partial y} + zt\frac{\partial}{\partial z} + t^2\frac{\partial}{\partial t} - \left(\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} + \frac{3t}{2}\right) u\frac{\partial}{\partial u}$$

With help of the Lie series

$$(x, y, z, t, u) \to \exp(\varepsilon K)(x, y, z, t, u) \tag{9}$$

where K is an infinitesimal generator and ε the group parameter, we obtain the corresponding transformation group. By a straightforward calculation we find

$$
\begin{array}{llllll}
X: & x\to x+\varepsilon, & y\to y, & z\to z, & t\to t, & u\to u \\
Y: & x\to x, & y\to y+\varepsilon, & z\to z, & t\to t, & u\to u \\
Z: & x\to x, & y\to y, & z\to z+\varepsilon, & t\to t, & u\to u \\
T: & x\to x, & y\to y, & z\to z, & t\to t+\varepsilon, & u\to u \\
V: & x\to x, & y\to y, & z\to z, & t\to t, & u\to e^{\varepsilon}u \\
S: & x\to e^{\varepsilon}x, & y\to e^{\varepsilon}x, & z\to e^{\varepsilon}z, & t\to e^{2\varepsilon}t, & u\to u \\
G_1: & x\to x+\varepsilon t, & y\to y, & z\to z, & t\to t, & u\to u\,e^{-t\varepsilon^2/4-x\varepsilon/2} \\
G_2: & x\to x, & y\to y+\varepsilon t, & z\to z, & t\to t, & u\to u\,e^{-t\varepsilon^2/4-y\varepsilon/2} \\
G_3: & x\to x, & y\to y, & z\to z+\varepsilon t, & t\to t, & u\to u\,e^{-t\varepsilon^2/4-z\varepsilon/2}
\end{array}
\tag{10}
$$

$$R_{12}: \quad \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad z\to z,\ u\to u$$

$$R_{23}: \quad \begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \qquad x\to x,\ u\to u$$

$$R_{31}: \quad \begin{pmatrix} z \\ x \end{pmatrix} \to \begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} z \\ x \end{pmatrix}, \qquad y\to y,\ u\to u$$

$$P: \quad x\to\frac{x}{1-\varepsilon t},\quad y\to\frac{y}{1-\varepsilon t},\quad z\to\frac{z}{1-\varepsilon t},\quad t\to\frac{t}{1-\varepsilon t}$$

$$u\to\frac{u}{(1-\varepsilon t)^{3/2}}\exp\left[-\frac{\varepsilon}{4}\left(\frac{x^2}{1-\varepsilon t}+\frac{y^2}{1-\varepsilon t}+\frac{z^2}{1-\varepsilon t}\right)\right]$$

A further vector field which leaves three-dimensional diffusion equation invariant is given by U. We have described the properties of this vector field in Section 2. We can easily extend these properties to the three-dimensional case. In particular, this means that we find solutions to the three-dimensional diffusion equation via the commutators $[G_i, U]$ $(i = 1, 2, 3)$, $[P, U]$, $[G_i, [G_i, U]]$, and so on.

Let us now study two diffusion equations which include chemical reactions. The first equation under consideration is given by (Crank, 1975)

$$\partial u/\partial t = D\Delta u - ku \tag{11}$$

where D is the diffusion constant which is assumed to be constant, and k the rate constant of the chemical reaction. Δ is the three-dimensional Laplace operator. The chemical reaction under consideration is given by $U \xrightarrow{k} S$, where u denotes the concentration of the species U. The species U decays in the species S. We mention that equation (11) can also be used for describing the conduction of heat along a wire which loses heat from its surface at a rate proportional to its temperature. We also have the same equation when the species U undergoes radioactive decay.

The second equation under consideration is given by

$$\partial u/\partial t = D\Delta u + ks_0 \exp(-kt) \tag{12}$$

where the chemical reaction is of the form $S \xrightarrow{k} U$. s_0 denotes the concentration of the species S at time $t = 0$ and we assume that the concentration of U is equal to 0 at time $t = 0$. Moreover we assume that the species S does not diffuse through the medium, i.e., the diffusion constant for this species is equal to zero. If we assume that the concentration of U is equal to u_0 at time $t = 0$, then we must replace s_0 by $s_0 + u_0$.

We may well ask under which transformation groups the partial differential equations given above are invariant.

Consider now the patial differential equation (11) and the vector fields given by equation (8). We find that the vector fields

$$\{X, Y, Z, T, V, G_1, G_2, G_3, R_{12}, R_{23}, R_{32}\} \tag{13}$$

leave the diffusion equation (11) invariant. Equation (1) is no longer invariant under $\{S, P\}$, but the equation (11)

$$S^* = x\partial/\partial x + y\partial/\partial y + z\partial/\partial z + 2t\partial/\partial t - 2ktu\partial/\partial u \tag{14}$$

and

$$P^* = xt\partial/\partial x + yt\partial/\partial y + zt\partial/\partial z + t^2\partial/\partial t - t^2 ku\partial/\partial u$$
$$-(x^2/4 + y^2/4 + z^2/4 + 3t/2)u\partial/\partial u \tag{15}$$

respectively.

The diffusion equation (1) can be transformed with the help of the transformation

$$u(x, y, z, t) = u'(x, y, z, t) \exp(-kt) \tag{16}$$

into the diffusion equation

$$\partial u'/\partial t = D\Delta u' \tag{17}$$

With the help of this transformation we can also obtain the vector fields S^* and P^* from the vector fields S and P.

Consider now the diffusion equation (12). The partial differential equation is invariant under the vector fields

$$\{X, Y, Z, R_{12}, R_{23}, R_{31}\} \tag{18}$$

Moreover, equation (12) is invariant under

$$\begin{gathered}
\partial/\partial t + k\exp(-kt)\partial/\partial u \\
[u + s_0\exp(-kt)]\partial/\partial u \\
x\partial/\partial x + y\partial/\partial y + z\partial/\partial z + 2t\partial/\partial t + 2ks_0 t\exp(-kt)\partial/\partial u \\
t\partial/\partial x - x\{[u + s_0\exp(-kt)]/2\}\partial/\partial u \\
t\partial/\partial y - y\{[u + s_0\exp(-kt)]/2\}\partial/\partial u \\
t\partial/\partial z - z\{[u + s_0\exp(-kt)]/2\}\partial/\partial u \\
xt\partial/\partial x + yt\partial/\partial v + zt\partial/\partial z + t^2\partial/\partial t + kt^2\exp(-kt)\partial/\partial u \\
-(x^2/4 + y^2/4 + z^2/4 + 3t/2)[u + s_0\exp(-kt)]\partial/\partial u
\end{gathered} \tag{19}$$

The diffusion equation (12) can be transformed with the help of the transformation

$$u(x, y, z, t) = u'(x, y, z, t) - s_0\exp(-kt) \tag{20}$$

into the diffusion equation $\partial u'/\partial t = D\Delta u'$.

4. THE LIE ALGEBRA OF THE SYMMETRY GENERATORS

Consider now the properties of the vector fields given by equation (8). By a straightforward calculation we find

$$\begin{gathered}
[X, P] = G_1, \qquad [Y, P] = G_2, \qquad [Z, P] = G_3 \\
[V, P] = 0, \qquad [S, P] = 2P \\
[G_1, P] = [G_2, P] = [G_3, P] = 0 \\
[R_{12}, P] = [R_{23}, P] = [R_{31}, P] = 0 \\
[S, G_1] = G_1, \qquad [S, G_2] = G_2, \qquad [S, G_3] = G_3 \\
[G_1, R_{12}] = G_2, \qquad [G_1, R_{23}] = 0, \qquad [G_1, R_{31}] = G_3 \\
[G_2, R_{12}] = -G_1, \qquad [G_2, R_{23}] = G_3, \qquad [G_2, R_{31}] = 0 \\
[G_3, R_{12}] = 0, \qquad [G_3, R_{23}] = -G_2, \qquad [G_3, R_{31}] = G_1 \\
[G_1, G_2] = [G_1, G_3] = [G_2, G_3] = 0 \\
[S, R_{12}] = [S, R_{23}] = [S, R_{31}] = 0 \\
[V, G_1] = [V, G_2] = [V, G_3] = 0
\end{gathered} \tag{21}$$

Consequently, Abelian Lie algebras are given by

$$\{X, Y, Z, T\}, \qquad \{G_1, G_2, G_3, P\}, \qquad \{V, G_1, G_2, G_3\}$$
$$\{V, P\}, \qquad \{R_{12}, P\}, \qquad \{R_{23}, P\}, \qquad \{R_{31}, P\} \tag{22}$$
$$\{R_{12}, S\}, \qquad \{R_{23}, S\}, \qquad \{R_{31}, S\}$$

5. SIMILARITY SOLUTIONS

With the help of the symmetry groups given by equation (9) we are able to find so-called similarity solutions of the diffusion equation. This means, we can derive a so-called similarity variable, say, η, which depends on the time coordinate t and the space coordinates x, y, z. With the help of this similarity variable η we are now able to reduce the partial differential equation to an ordinary differential equation, where the independent variable of this ordinary differential equation is the similarity variable η. In the following we demonstrate the approach for two particular cases.

Consider the infinitesimal generators

$$\{R_{12}, Z, T + aV\} \tag{23}$$

where $a \in \mathbb{R}$. The vector fields R_{12}, Z, and $T + aV$ form a basis of an Abelian Lie algebra. We mention that we need three vector fields for obtaining an ordinary differential equation. With each vector field we can eliminate one independent variable. In the present case there are four independent variables. For finding the similarity variable and the ordinary differential equation which can be derived from the infinitesimal generators we need the corresponding transformation groups and the composition of these transformation groups.

According to the infinitesimal generators R_{12}, Z, and $T + aV$ we find

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \cos \varepsilon_1 & -\sin \varepsilon_1 \\ \sin \varepsilon_1 & \cos \varepsilon_1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \qquad z_1 = z_0, \qquad t_1 = t_0, \qquad u_1 = u_0$$
$$x_2 = x_1, \qquad y_2 = y_1, \qquad z_2 = z_1 + \varepsilon_2, \qquad t_2 = t_1, \qquad u_2 = u_1 \tag{24}$$
$$x_3 = x_2, \qquad y_3 = y_2, \qquad z_3 = z_2, \qquad t_3 = t_2 + \varepsilon_3, \qquad u_3 = u_2 \exp(a\varepsilon_3)$$

The composition of these transformation groups gives the three-parameter transformation group

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} \cos \varepsilon_1 & -\sin \varepsilon_1 \\ \sin \varepsilon_1 & \cos \varepsilon_1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{25a}$$

$$z_3 = z_0 + \varepsilon_2 \tag{25b}$$

$$t_3 = t_0 + \varepsilon_3 \tag{25c}$$

$$u_3 = u_0 \exp(a\varepsilon_3) \tag{25d}$$

In the following we put $x_3 = x$, $y_3 = y$, $z_3 = z$, $t_3 = t$, and $u_3 = u$. We choose $x_0 = \eta$ (η = similarity variable), $y_0 = z_0 = t_0 = 0$. Then the above equations can be solved with respect to ε_1, ε_2, ε_3, and η and we find

$$\varepsilon_1 = \arctan\left(\frac{y}{x}\right), \qquad \varepsilon_2 = z, \qquad \varepsilon_3 = t \tag{26}$$

The similarity variable η takes the form

$$\eta = (x^2 + y^2)^{1/2} \tag{27}$$

Taking into account the equation (25d) we obtain the ansatz

$$u(x, y, z, t) = \bar{u}(\eta) \exp(at) \tag{28}$$

Inserting this ansatz into the diffusion equation we find an ordinary equation where the independent variable is given by η and the dependent variable is given by $\bar{u}$. By a straightforward calculation we obtain the following ordinary differential equation:

$$\frac{d^2\bar{u}}{d\eta^2} + \frac{1}{\eta} \cdot \frac{d\bar{u}}{d\eta} = a\bar{u} \tag{29}$$

The resulting ordinary differential equation is of Bessel's type and can be solved with the help of Bessel functions.

6. LIE–BÄCKLUND TRANSFORMATION GROUPS

For further investigations, in particular for obtaining Lie–Bäcklund transformation groups which leave the diffusion equation invariant, we consider our partial differential equation within the jet bundle formalism (Johnson, 1962; Olver, 1979; Steeb et al., 1982). Since most physicists are not familiar with this formalism we give a short review.

First of all let us introduce the notation. A triple (N, π, M) is called a fibered manifold if M and N are differentiable manifolds and $\pi : N \to M$ is a surjective submersion. The so-called base manifold M represents the independent variables. In most cases in physics $M = \mathbb{R}^4$ or an open subset of $\mathbb{R}^4$. The manifold N represents the dependent variables (the fields) and the independent variables. In most cases in physics N will be an open subset of the Euclidian space $\mathbb{R}^4 \times \mathbb{R}^n$. Now let $\dim M = m$ and $\dim N = n + m$ and let (x_i, u_j) $(1 \le i \le m,\ 1 \le j \le n)$ denote the coordinate function defined by a fiber chart. Sections of N are defined as smooth maps $s : M \to N$ such that $\pi \circ s = 1_M$, where 1_M is the identity map of M. We call the functions (x_i, u_j) the fiber coordinates on N. The r-jet bundle $J^r(N)$ is given by the equivalence classes of sections of N having rth order contact. The coordinate functions on $J^r(N)$ are denoted by $(x_i, u_j, u_{ji}, u_{ji_1i_2}, \ldots, u_{ji_qi_2\ldots i_r})$, where

$$i, i_1, \ldots, i_r \in \{1, \ldots, m\}, \qquad j \in \{1, \ldots, n\} \quad \text{and} \quad 1 \le i_1 \le i_2 \ldots \le i_r \le m.\ u_{ji_1\ldots i_p}$$

The quantity $u_{ij_1\ldots j_n}$ corresponds to the partial derivative of u_j with respect to $x_{i_1} \ldots x_{i_p}$. The infinite jet bundle is denoted by $J(N)$. Within the jet bundle formalism a system of partial differential equations of order r is defined to be a submanifold of $J^r(N)$. Consider a system of partial differential equations of order r

$$F_\nu(x_i, u_j, \partial u_j/\partial x_i, \ldots, \partial^r u_j/\partial x_{i_1}\partial x_{i_2} \ldots \partial x_{i_r}) = 0 \qquad (\nu = 1, \ldots, q) \tag{30}$$

Within the jet bundle formalism we consider the submanifold

$$F_\nu(x_i, u_j, u_{ji}, \ldots, u_{ji_1\ldots i_r}) = 0 \tag{31}$$

and the contact forms

$$\begin{aligned} \theta_j &= du_j - \sum_{i=1}^{m} u_{ji}\, dx_i \\ &\vdots \\ \theta_{ji_1\ldots i_r} &= du_{ji_1\ldots i_{r-1}} - \sum_{k=1}^{m} u_{ji_1\ldots i_r k}\, dx_k \end{aligned} \tag{32}$$

Consider now the vector field D_i defined on $J(N)$ by

$$D_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} u_{ji} \frac{\partial}{\partial u_j} + \cdots + \sum_{j=1}^{n} \sum_{i_1,\ldots,i_r=1}^{m} u_{jii_1\ldots i_r} \frac{\partial}{\partial u_{ji_1,\ldots,i_r}} + \cdots \tag{33}$$

The summation on the right-hand side is restricted to $1 \le i_1 \le i_2 \le \cdots \le i_r \le m$. D_i is sometimes called the operator of total differentiation. Together with equation (31) we consider all differential consequences $D_i F_\nu = 0, \ldots, D_{i_1} D_{i_2} \ldots F_\nu = 0$. Let $\Omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$ be the volume form on M. x_m will play the role of the time coordinate.

Definition. The $(m-1)$ form

$$\omega = \sum_{k=1}^{m} f_k(x_i, u_j, u_{ji}, \ldots) \left(\frac{\partial}{\partial x_k} \rfloor \Omega \right) \tag{34}$$

defined on $J(N)$ is called a conservation law of equation (30) if $(js)^*(d\omega) = 0$ whenever $s: M \to N$ is a solution to equation (30). js is the jet extension of s up to infinite order. $\partial/\partial x_k \rfloor \Omega$ denotes the contraction.

Another possibility for defining conserved currents is: The $(m-1)$ form ω given above is called a conservation law if $d\omega \in J$, where J is the differential ideal generated by F_ν, D_iF_ν, ..., and the contact forms. We mention that the first definition is the more general one.

For deriving conserved currents we consider the vector fields

$$Z=\sum_{i=1}^{m} a_i\partial/\partial x_i+\sum_{j=1}^{n} b_j\partial/\partial u_j \tag{35}$$

where a_i and b_j depend upon $(x_i, u_j, u_{ji}, \ldots)$. The corresponding vertical vector field is given by

$$Z_V=\sum_{j=1}^{n}\left(b_j-\sum_{i=1}^{m} a_i u_{ji}\right)\frac{\partial}{\partial u_j} \tag{36}$$

We denote by $\bar{Z}_V$ the prolongation of the vector field Z_V up to infinite order.

Definition. The system of partial differential equations (30) which is described within the jet bundle formalism by equation (31) and the contact forms is called invariant under the vector field $\bar{Z}_V$ if

$$L_{\bar{Z}_V}F_\nu \triangleq 0 \tag{37}$$

where $\triangleq$ stands for the restriction to solutions to equation (30).

Again we can give a definition which is not so general, but frequently used. Here the system of partial differential equations is called invariant if $L_{\bar{Z}}F_\nu\in J$, $L_{\bar{Z}}\theta_j\in J, \ldots$, where J is the differential ideal generated by F_ν, D_iF_ν, and the contact forms.

Assume that the vector field $\bar{Z}_V$ is integrable to the corresponding group action $u\to\exp(\varepsilon\bar{Z}_V)u$. Then, owing to invariance, a solution $s:M\to N$ is carried into a new solution $\exp(\varepsilon\bar{Z}_V)s$.

Theorem. Assume that the system of partial differential equations (30) is invariant under the vector field $\bar{Z}_V$. Let ω be a conservation law of equation (30). Then $L_{\bar{Z}_V}\omega$ is also a conservation law of equation (30).

The proof is by straightforward calculation. (cf. also Steeb and Strampp, 1982).

Let us now prove the theorem given in Section 2. The prolongation of the vector field $A=f(x,t)\partial/\partial u$ up to second order is given by

$$\bar{A}=A+\frac{\partial f}{\partial t}\cdot\frac{\partial}{\partial u_t}+\frac{\partial f}{\partial x}\cdot\frac{\partial}{\partial u_x}+\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial}{\partial u_{xx}}+\frac{\partial^2 f}{\partial t^2}\cdot\frac{\partial}{\partial u_{tt}}+\frac{\partial^2 f}{\partial x\partial t}\cdot\frac{\partial}{\partial u_{xt}} \tag{38}$$

It follows that

$$\bar{A}(u_t-u_{xx})=\frac{\partial f}{\partial t}-\frac{\partial^2 f}{\partial x^2}=0 \tag{39}$$

since f satisfies the diffusion equation.

7. CONSERVED CURRENTS OF THE DIFFUSION EQUATION

With the help of the theorem described in Section 6 we are able to derive conserved currents with the help of the infinitesimal generator given by equation (25). The diffusion equation can be written as

$$js^*(d\omega) = 0 \tag{40}$$

where

$$\omega = u\, dx \wedge dy \wedge dz + u_x\, dt \wedge dy \wedge dz + u_y\, dt \wedge dz \wedge dx + u_z\, dt \wedge dx \wedge dy \tag{41}$$

Consequently, equation (40) is a conservation law. As a consequence it follows that

$$Q(u) = \int_{\mathbb{R}^3} u(x, y, z, t)\, dx\, dy\, dz \tag{42}$$

is a conserved quantity. Q is the total amount of the diffusing substance. Taking the Lie derivative of ω with respect to the vector fields given by equation (8) we can find further conservation laws. The vector fields X, Y, Z, T, V, R_{12}, R_{23}, and R_{31} do not give new conservation laws. In this case we obtain ω or zero by taking the Lie derivative of ω with respect to these vector fields. On the other hand we find a hierarchy of conservation laws (and therefore a hierarchy of constants of motion) when we consider the vector fields G_1, G_2, G_3, and P.

Consider now the vector field symmetry generators G_1 and the differential form (the conservation law) ω. By Lie derivative of the differential form ω with respect to the symmetry generator $\bar{G}_1$ we find

$$\begin{aligned} L_{\bar{G}_1}\omega = &-\frac{xu}{2}\, dx \wedge dy \wedge dz - \frac{u_z x}{2}\, dt \wedge dx \wedge dy \\ &+ \left(\frac{u}{2} - \frac{u_x x}{2}\right) dt \wedge dy \wedge dz - \frac{u_y x}{2}\, dt \wedge dz \wedge dx \end{aligned} \tag{43}$$

From this expression it follows that

$$P_1(u) = \int_{\mathbb{R}_3} xu(x, y, z, t)\, dx\, dy\, dz \tag{44}$$

is a conserved quantity. Consequently, for actual calculation of the quantity P_1 we can insert the initial distribution $\phi(x, y, z) = u(x, y, z, t = 0)$ into equation (44). For example, if ϕ is an even function with respect to each coordinate we find that

$$\int_{\mathbb{R}^3} x\phi(x, y, z)\, dx\, dy\, dz = 0 \tag{45}$$

When we calculate the Lie derivative of the differential form $L_{\bar{G}_1}\omega$ with respect to $\bar{G}_1$ we find a further conservation law and therefore a constant of motion. A straightforward calculation yields that

$$P_2(u) = \int_{\mathbb{R}^3} \left(-\frac{t}{2} + \frac{x^2}{4} \right) u(x, y, z, t)\, dx\, dy\, dz \tag{46}$$

is a conserved quantity. Now we can expand this approach up to infinite order. $P_n(u)$ is given as follows. Let f_n be the function

$$f_n(x, y, z, t) = \left(\frac{x}{2} - t\frac{\partial}{\partial x} \right)^n \cdot 1 \tag{47}$$

Then $P_n(u)$ is given by

$$P_n(u) = \int_{\mathbb{R}^3} u(x, y, z, t) f_n(x, y, z, t)\, dx\, dy\, dz \tag{48}$$

For the vector fields G_2, G_3, and P we also obtain a hierarchy of conservation laws.

8. NONLINEAR DIFFUSION EQUATIONS AND LINEARIZATION

Nonlinear diffusion equations arise when we study concentration-dependent diffusion. The equation under consideration is then

$$\frac{\partial v}{\partial t} = \operatorname{div}(D(v) \operatorname{grad} v) \tag{49}$$

In this section we study a class of nonlinear diffusion equations and its connection with the linear diffusion equation. The class of nonlinear diffusion equations is given in such a manner that there is a transformation (of course nonlinear) which linearizes the nonlinear diffusion equation. In the one-dimensional case several authors (Ames, 1965; Kaup, 1980; Bluman and Kumei, 1980; Munier et al., 1981; Ibragimov and Shabat, 1980; Bluman, 1980) have studied the problem of linearizing nonlinear diffusion equations. In the literature the best known example is the so-called Burgers equation (Kaup, 1980)

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v\frac{\partial v}{\partial x} \tag{50}$$

Inserting the nonlinear transformation

$$v(x, t) = -2\frac{\partial}{\partial x}\ln u(x, t) \equiv -2\frac{(\partial u/\partial x)(x, t)}{u(x, t)} \tag{51}$$

into the Burgers equation we find that u satisfies the linear diffusion equation $\partial u/\partial t = \partial^2 u/\partial x^2$.

The transformation given by equation (51) is sometimes called a Bäcklund transformation since a derivative of u appears on the right-hand side. The Cauchy initial problem is solved for the linear diffusion equation, i.e., find u satisfying $\partial^2 u/\partial x^2 = \partial u/\partial t$ such that $u(x, 0) = \phi(x)$. Therefore with the help of a nonlinear transformation we can solve the Cauchy problem for the Burgers equation; but the calculation shows that for the Burgers equation the initial perturbation must obey a restrictive condition in order that the solution exist. The Burgers equation (50) can be written in the form of a conservation law, namely,

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2 - \frac{\partial v}{\partial x}\right) = 0 \tag{52}$$

Since the transformation given by (51) is not invertible we are not able to find conservation laws of the Burgers equation from conservation laws of the linear diffusion equation.

Let us assume that the nonlinear transformation which linearizes the nonlinear diffusion equation is invertible.

Since solutions and conserved currents of the linear diffusion equation are known we are able to derive solutions and conserved currents of the nonlinear diffusion equations which are associated with the linear diffusion equation (1) via nonlinear transformations. In the following we assume that the quantities x, y, z, t, u are given so that they are dimensionless.

Let us now consider the three-dimensional case. First of all we discuss two examples. Consider first the nonlinear diffusion equation

$$\frac{\partial v}{\partial t} = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \tag{53}$$

With the help of the nonlinear transformation

$$v(x, y, z, t) = \ln u(x, y, z, t) \tag{54}$$

the nonlinear equation can be linearized. The transformation is invertible and we have $u(x, y, z, t) = \exp[v(x, y, z, t)]$. Consequently, we can write the nonlinear equation as a conservation law, namely,

$$\frac{\partial e^v}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial e^v}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial e^v}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial e^v}{\partial z}\right) = 0 \tag{55}$$

Thus we can transform solutions, symmetry generators, and conservation laws from the linear diffusion equation to the nonlinear diffusion equation. Let u_1 and u_2 be solutions to the linear diffusion equation. Then u_1+u_2 is also a solution to the linear diffusion equation. Now let v_1 and v_2 be solutions to the nonlinear diffusion (53). We may well ask whether the two solutions can be combined so that we find a new solution (so-called nonlinear superposition). With the help of the transformation (54) we can easily find the nonlinear superposition. Starting from $u_1+u_2=\exp(v)$, $u_1=\exp(v_1)$, and $u_2=\exp(v_2)$ we obtain $\ln(u_1+u_2)=v$ and therefore $v=\ln[\exp(v_1)+\exp(v_2)]$ is a solution to the nonlinear diffusion equation (53).

Another example of a nonlinear diffusion equation which can be linearized is the following:

$$\frac{\partial v}{\partial t}=-\frac{1}{v}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} \tag{56}$$

The linear diffusion equation can be obtained via the transformation

$$v(x, y, z, t)=\exp[u(x, y, z, t)] \tag{57}$$

Consequently, we can write equation (56) as a conservation law

$$\frac{\partial(\ln v)}{\partial t}=\frac{\partial}{\partial x}\left(\frac{\partial \ln v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \ln v}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial \ln v}{\partial z}\right) \tag{58}$$

The nonlinear superposition of two solutions v_1 and v_2 to the equation (56) can be found as described above. Starting from $u_1+u_2=\ln v$, $u_1=\ln v_1$ and $u_2=\ln v_2$ we obtain $v=\exp(u_1+u_2)$ and therefore $v=\exp(\ln v_1+\ln v_2)$. Consequently, $v=v_1v_2$. Thus if v_1 and v_2 are solutions to the equation (56), then $v=v_1v_2$ is also a solution to equation (56). Since the nonlinear transformation is invertible we are able to find symmetry generators of the equation (56) from the symmetry generators of the linear diffusion equation [equation (8)].

Let us now generalize the results given above. Consider a smooth function f of v which is invertible in the region under consideration. Now we write

$$\frac{\partial f(v)}{\partial t}-\frac{\partial}{\partial x}\cdot\frac{\partial f(v)}{\partial x}-\frac{\partial}{\partial y}\cdot\frac{\partial f(v)}{\partial y}-\frac{\partial}{\partial z}\cdot\frac{\partial f(v)}{\partial z}=0 \tag{59}$$

By a straightforward calculation it follows that

$$f'\frac{\partial v}{\partial t}-f'\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-f''\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]=0 \tag{60}$$

where the prime denotes the derivative of f with respect to v. Since f is invertible, we find that

$$\frac{\partial v}{\partial t}-\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\frac{f''}{f'}\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right]=0 \qquad (61)$$

The nonlinear diffusion equation (61) can be linearized with the help of the function f^{-1}. Since we know the solutions, symmetry generators, and conservation laws of the linear diffusion equation (1) we find with the help of the function f and f^{-1} solutions, symmetry generators and conservation laws of the nonlinear diffusion equation (61).

A natural question is what happens when we also include derivatives of v in our transformation. This means the function should depend on u, u_x, u_y, u_z and therefore we have a Bäcklund transformation. The problem is that in most cases the function f is not invertible.

However, the following extension is possible. Let

$$\Delta v+f(v)(\text{grad } v)^2+a(x,t)\text{grad } v+b(x,t)\partial v/\partial t=0 \qquad (62)$$

where $x=(x_1,\ldots,x_n)^T$ and $a=(a_1,\ldots,a_n)$. f, b, and a_i are given smooth functions. With the help of the transformation

$$u=\int_{v_0}^{v}\left\{\exp\left[\int_{v_0}^{\beta} f(\alpha)\,d\alpha\right]\right\}d\beta \qquad (63)$$

we obtain the linear partial differential equation

$$\Delta u+a(x,t)\text{ grad } u+b(x,t)\partial u/\partial t=0 \qquad (64)$$

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